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LETTER TO THE EDITOR

Chaotic systems and maximum entropy formalism

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Abstract. We show how the maximum entropy formalism can be applied in nonlinear chaotic systems. In particular we show how one can use a few moments of the time evolution of a dynamical variable to infer the probability density and hence the Ljapunov exponent.

The maximum entropy principle is a powerful tool in the investigations of image reconstruction, spectral analysis, seismic inversion, inverse scattering, etc. It is proven to be the only consistent method for inferring from incomplete information. Here we show how the maximum entropy formalism can be applied to the study of chaotic systems. For the sake of simplicity we consider one-dimensional maps $f : [0, 1] \to [0, 1]$. We assume that the map is chaotic and ergodic. The mathematical foundation of one-dimensional chaotic and ergodic maps is well described in literature [1–5]. The one-dimensional map can also be written as a difference equation

$$x_{t+1} = f(x_t) \tag{1}$$

where $t = 0, 1, 2, \ldots$ and $x_0 \in [0, 1]$. Let us now introduce the quantities useful in the study of chaotic systems. The nth moment of the time evolution of x_t is defined as

$$\langle x_t^n \rangle_T := \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} x_t^n \tag{2}$$

where $n = 1, 2, \ldots$. Obviously the moments depend on the initial conditions. For $n = 1$ we obtain the time average. We find for almost all initial conditions the same value. In the case of ergodic systems the moments and the probability density are related by

$$\langle x_t^n \rangle = \int_0^1 y^n \rho(y)\, \mathrm{d}y \tag{3}$$

where $\rho > 0$ for $x \in [0, 1]$ and

$$\int_0^1 \rho(x)\, \mathrm{d}x = 1. \tag{4}$$

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The time evolution of the probability density is given by

$$\rho_{t+1}(x) = \int_0^1 \rho_t(y)\delta(x - f(y))\,\mathrm{d}y \tag{5}$$

where $t = 0, 1, 2, \ldots$ and ρ_0 is the initial probability density, for example $\rho_0 = 1$. The asymptotic probability density is obtained by solving the Frobenius–Perron integral equation

$$\rho(y) := \int_0^1 \rho(x)\delta(y - f(x))\,\mathrm{d}x. \tag{6}$$

For some maps we can solve the Frobenius–Perron integral equation exactly. An example is the logistic map $f(x) = 4x(1 - x)$. In general we can only find the probability density approximately. A numerical method is described by Steeb [6]. If we know the probability density we can evaluate the Ljapunov exponent λ as

$$\lambda := \int_0^1 \rho(x) \ln\left|\frac{\mathrm{d}f}{\mathrm{d}x}\right| \mathrm{d}x. \tag{7}$$

The Ljapunov exponent can also be calculated as follows: The variational equation of $x_{t+1} = f(x_t)$ is given by

$$y_{t+1} = \frac{\mathrm{d}f}{\mathrm{d}x}(x \to x_t)y_t \tag{8}$$

where $t = 0, 1, 2, \ldots$. Then the Ljapunov exponent is given by

$$\lambda = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \ln\left|\frac{y_{t+1}}{y_t}\right|. \tag{9}$$

One can use the maximum entropy formalism [7–11] to obtain the probability density approximately using as information N moments. In our examples we consider the case $N = 2$. The missing information function (entropy) of a probability density ρ is defined by

$$I = -\int_0^1 \rho(x) \ln \rho(x). \tag{10}$$

In the maximum entropy formalism, one maximizes the missing information subject to the constraints of the available information and to the normalization of the probability density. In our case we assume that we have the N lowest moments of the time evolution of the dynamical variables which, for ergodic systems, are equal to the moments of the temporal probability density. The constraints are introduced via the method of Lagrange multipliers $\lambda_0, \lambda_1, \ldots, \lambda_N$. Our aim is to find the approximate probability density ρ_{app} which minimizes

$$I' = -\int_0^1 \rho_{\mathrm{app}} \ln \rho_{\mathrm{app}}\,\mathrm{d}x + \lambda_0\left(1 - \int_0^1 \rho_{\mathrm{app}}\,\mathrm{d}x\right) + \sum_{n=1}^{N} \lambda_n\left(\langle x^n\rangle - \int_0^1 x^n \rho_{\mathrm{app}}\,\mathrm{d}x\right) \tag{11}$$

where λ_n, $n = 0, 1, \ldots, N$ are the Lagrange multipliers for the $N+1$ constraints. Performing the minimization we obtain

$$\rho_{\mathrm{app}}(x) = \exp\left(-1 - \sum_{n=0}^{N} \lambda_n x^n\right) \equiv \frac{1}{Z}\exp\left(-\sum_{n=1}^{N} \lambda_n x^n\right) \tag{12}$$

where $Z := \mathrm{e}^{1+\lambda_0}$ is determined from the normalization of the probability density so that

$$1 = \frac{1}{Z}\int_0^1 \exp\left(-\sum_{n=1}^{N}\lambda_n x^n\right)\mathrm{d}x. \tag{13}$$

The remaining Lagrange multipliers are obtained by solving the following set of N coupled nonlinear equations for λ_m, $m = 1, \ldots, N$

$$\langle x^m\rangle = \frac{1}{Z}\int_0^1 x^m \exp\left(-\sum_{n=1}^{N}\lambda_n x^n\right)\mathrm{d}x \qquad m = 1, 2, \ldots, N. \tag{14}$$

Let us now consider two examples. For both we can solve the dynamical system $x_{t+1} = f(x_t)$ exactly. As a first example we consider the tent map $f : [0, 1] \to [0, 1]$

$$f(x) = \begin{cases} 2x & \text{for } x \in [0, 1/2] \\ 2(1-x) & \text{for } x \in [1/2, 1] \end{cases} \tag{15}$$

which is chaotic and ergodic (even mixing). The (exact) probability density is given by $\rho(x) = 1$. Furthermore the moments take the form

$$\langle x_t^n\rangle = \frac{1}{n+1}. \tag{16}$$

Let us now assume we know the first two moments $\langle x_t\rangle = 1/2$ and $\langle x_t^2\rangle = 1/3$. Then the approximate probability density is given by

$$\rho_{\mathrm{app}} = \frac{1}{Z}\exp(-\lambda_1 x - \lambda_2 x^2) \tag{17}$$

and we have to solve the following nonlinear system of equations

$$1 = \int_0^1 \exp(-1 - \lambda_0 - \lambda_1 x - \lambda_2 x^2)\,\mathrm{d}x \equiv I(\lambda_0, \lambda_1, \lambda_2) \tag{18a}$$

$$\frac{1}{2} = \int_0^1 x\exp(-1 - \lambda_0 - \lambda_1 x - \lambda_2 x^2)\,\mathrm{d}x = -\frac{\mathrm{d}I}{\mathrm{d}\lambda_1} \tag{18b}$$

$$\frac{1}{3} = \int_0^1 x^2\exp(-1 - \lambda_0 - \lambda_1 x - \lambda_2 x^2)\,\mathrm{d}x = -\frac{\mathrm{d}I}{\mathrm{d}\lambda_2} \tag{18c}$$

to find the Lagrange multipliers λ_0, λ_1 and λ_2. Obviously the solution is $\lambda_0 = -1$, $\lambda_1 = 0$, $\lambda_2 = 0$. Thus the approximate probability density is given by $\rho_{\mathrm{app}}(x) = 1$. Thus the exact probability density and the approximate probability density coincide. We notice that the integration of (18a) leads to the error function.

As a second example we consider the logistic map $f(x) = 4x(1-x)$. The probability density is given by

$$\rho(x) = \frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}. \tag{19}$$

The moments take the form

$$\langle x_t^n \rangle = \frac{1}{2^{2n}} \binom{2n}{n}. \tag{20}$$

Thus the first and second moments are given by $\langle x_t \rangle = \frac{1}{2}, \langle x_t^2 \rangle = \frac{3}{8}$. We have to solve

$$1 = \int_0^1 \exp(-1 - \lambda_0 - \lambda_1 x - \lambda_2 x^2)\, \mathrm{d}x \tag{21a}$$

$$\frac{1}{2} = \int_0^1 x \exp(-1 - \lambda_0 - \lambda_1 x - \lambda_2 x^2)\, \mathrm{d}x \tag{21b}$$

$$\frac{3}{8} = \int_0^1 x^2 \exp(-1 - \lambda_0 - \lambda_1 x - \lambda_2 x^2)\, \mathrm{d}x. \tag{21c}$$

We solved system (21) numerically and find

$$\lambda_0 = 2.692\,42 \qquad \lambda_1 = -6.768\,25 \qquad \lambda_2 = -\lambda_1. \tag{22}$$

Obviously $\lambda_1 = -\lambda_2$ since ρ_{app} must be symmetric with respect to $x = 1/2$. Using these values we can find an approximation of the Ljapunov exponent using (7). Numerical integration of (7) with the approximate probability density yields $\lambda = 0.72$, which agrees very well with the exact value $\lambda = \ln 2$, if we take into account that we only considered two moments.

If we study a dynamical system with unknown moments we have to determine them numerically using (2) with a sufficiently large T. Then we insert the moments into the equations for the Lagrange multiplier and solve this set of equations numerically. Finally we determine the Ljapunov exponent numerically. A software program in C++ is available from the authors to solve the nonlinear equations for the Lagrange multiplier.

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